All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a finite group. Explain what is meant by the order, $\mathrm{ord}(g)$, of $g \in G$.

   If x is a generator of the cyclic group C_n of order n, show that, for $1 \leqslant m \leqslant n-1$,

   $$\mathrm{ord}(x^m) = \frac{n}{\mathrm{HCF}(m,n)}.$$

   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. Derive a necessary and sufficient condition on m for φ_m to be an automorphism.

   Describe explicitly

   (i) $\mathrm{Aut}(C_{15})$ as a product of cyclic groups;

   (ii) all homomorphisms $h : C_6 \to \mathrm{Aut}(C_{15})$.

2. Let K, Q be groups. If $h : Q \to \mathrm{Aut}(K)$ is a group homomorphism, explain what is meant by the *semi-direct product* $K \rtimes_h Q$.

   Let K, Q be subgroups of coprime order in a finite group G and suppose also that $K \lhd G$ and $|G| = |K||Q|$. Prove that G is isomorphic to a semi-direct product

   $$G \cong K \rtimes_h Q.$$

   Let G be a group of order 55 ; show that G has a normal subgroup of order 11. Hence classify all groups of order 55 up to isomorphism, giving a set of generators and relations for each isomorphism type.

   (You may use Sylow's Theorem without proof provided it is correctly stated.)

3. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X. Explain what is meant by

i) the orbit $\langle x \rangle$ of $x \in X$; ii) the stability subgroup G_x of $x \in X$.

Show that

a) for any $x \in X$ there exists a bijection $G/G_x \longleftrightarrow \langle x \rangle$ and

b) if $\langle x \rangle \cap \langle y \rangle \neq \emptyset$ then $\langle x \rangle = \langle y \rangle$.

Explain what is meant by the *class equation* of such an action (in each of its forms).

Let $Q(12)$ denote the quaternion group of order 12 given by the presentation

$$Q(12) = \langle x, y \mid x^3 = y^4 = 1, \ yx = x^2 y \rangle.$$

Describe the various forms of the class equation explicitly when $X = G = Q(12)$ and the action is *conjugation*

$$\circ \ : \ Q(12) \times Q(12) \to Q(12) \ ; \ g \circ h = ghg^{-1}.$$

4. Let G be a finite group acting on a finite set X. Define the *fixed point set* X^G.

State without proof a relationship which holds between $|X|$ and $|X^G|$ when $|G| = p^n$ for some prime p.

Let P, Q be subgroups of a group G; explain what is meant by saying that P *normalizes* Q. Show that, when P normalizes Q, there is a group isomorphism

$$PQ/Q \cong P/(P \cap Q).$$

Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, show that

$$N_p \equiv 1 \pmod{p}.$$

Let G be a group with $|G| = 91$. Stating any subsidiary results used show that

$$G \cong C_{91}.$$

5. Let A be a commutative integral domain containing a subfield $\mathbb{F}$ such that $\dim_{\mathbb{F}}(A)$ is finite. Show that A is a field.

   Hence show that if $p(x)$ is an irreducible polynomial of degree $n \geqslant 1$ over a field $\mathbb{F}$ then $\mathbb{F}[x]/(p(x))$ is a field .

   Let $\mathbb{F}_5$ denote the field with five elements ;

   i) show that both $x^2 + x + 2$ and $x^2 + 4x + 2$ are irreducible over $\mathbb{F}_5$ and

   ii) describe an explicit isomorphism of fields

$$L : \mathbb{F}_5[x]/(x^2 + x + 2) \xrightarrow{\simeq} \mathbb{F}_5[x]/(x^2 + 4x + 2).$$

6. State and prove Eisenstein's Criterion.

   Hence or otherwise show that the polynomial

$$a(x) = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 20$$

   is ireducible over $\mathbb{Q}$.

   Give the complete factorisations of the polynomials below into monic irreducible factors over $\mathbb{Q}$, justifying your answer in each case.

   i) $x^{10} - 1$;

   ii) $x^{10} + 1$;

   iii) $x^{20} - 3x^{10} - 4$.